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Note

The maximum sum of degrees above a threshold in planar graphs

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Abstract

We consider the maximum sum of the degrees of vertices with degree at least k in any simple planar graph on n vertices. West and Will solved this for $k \geqslant 12$ and $n \geqslant m_k$ and gave bounds for $6 \leqslant k \leqslant 11$. Here we complete the solution by constructing graphs achieving the bounds for all $n \geqslant m_k$ when $k \leqslant 11$.

1. Introduction

Given $n > k > 0$, consider the set of all simple planar graphs G on n vertices. Erdős and Griggs introduced $a_k(n) = \min_G |\{v : \deg(v) < k\}|$, the minimum number of vertices of degree less than k in such G. Except for a few small values of n, $a_k(n) = 0$ for $k < 6$. For $k = 6$ and $n \geqslant n_6$, $a_k(n)$ can be deduced from earlier work of Grünbaum and Motzkin [2]. For $k \geqslant 12$ and $n \geqslant n_k$, $a_k(n)$ was determined by West and Will [4]. The remaining cases, $7 \leqslant k \leqslant 11$, are the most complicated. West and Will found a lower bound for all large n and verified it is sharp for infinitely many n. In our earlier paper [1] (see also [3] for further work), we gave many constructions in order to achieve the lower bounds for all sufficiently large n. Let us define the notation

$$\Gamma_k(n) = \begin{cases} \dfrac{(k-6)n+12}{k-3} & \text{if } 6 \leqslant k \leqslant 10, \\ \dfrac{(k-6)n+18}{k-3} & \text{if } k = 11, \\ \dfrac{(k-8)n+16}{k-6} & \text{if } k \geqslant 12. \end{cases}$$

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Then the combined general result for $a_k(n)$ is the following.

Theorem 1 (Griggs and Lin [1]). *For positive integers k, there exists an n_k such that for all $n \geqslant n_k$,*

$$a_k(n) = \begin{cases} 0 & \text{if } k < 6, \\ 4 \text{ (resp., } 5) & \text{if } k = 6 \text{ and } n \text{ is even (resp., odd)}, \\ \lceil \Gamma_k(n) \rceil & \text{if } k \geqslant 7. \end{cases}$$

It is important to note that the condition $n \geqslant n_k$ is necessary here, since the formula fails for given k when n is small. We currently require $n_k = 14$ for $k < 6$, $n_6 = 8$, $n_7 = 52$, $n_8 = 27$, $n_9 = 244$, $n_{10} = 241$, $n_{11} = 92$, and $n_k = 3k - 10$ for $k \geqslant 12$.

In a similar way, in this paper, we present constructions of planar graphs to complete the solution of a related problem about sums of degrees. Whereas Theorem 1 is equivalent to maximizing the *number* of vertices of degree $\geqslant k$, Erdős and Vince asked about maximizing the *sum of degrees* of vertices of degree $\geqslant k$. Let

$$s_k(n) = \max_G \sum \{\deg(w) : \deg(w) \geqslant k\}.$$

We need to define

$$W_k(n) = \begin{cases} 3n - 12 + 3\left\lfloor \dfrac{3n-12}{k-3} \right\rfloor & \text{if } 7 \leqslant k \leqslant 10, \\ 3n - 18 + 3\left\lfloor \dfrac{3n-18}{8} \right\rfloor & \text{if } k = 11, \\ 2n - 16 + 6\left\lfloor \dfrac{2n-16}{k-6} \right\rfloor & \text{if } k \geqslant 12. \end{cases}$$

West and Will proved that $s_k(n) = W_k(n)$ for $k \geqslant 12$ and $n \geqslant 3(k-6)+8$. For $6 \leqslant k \leqslant 10$, they proved that $W_k(n)$ is an upper bound on $s_k(n)$ for n sufficiently large ($n \geqslant 4(k+6)/(12-k)$), and that equality holds for one residue class for each k. For $k = 11$, West and Will [4, Lemma 3.3] proved that $s_{11}(n) \leqslant 3n - 18 + 3m$ if $m \geqslant \frac{1}{3}(n-2)$, where $m = n - a_{11}(n)$. Therefore, $W_{11}(n)$ is an upper bound on $s_{11}(n)$ if $n \geqslant 92$. Using the constructions of our previous paper as well as some new ones, we complete the determination of $s_k(n)$ for all k and sufficiently large n.

Theorem 2. *For positive integers k, there exists an m_k such that for all $n \geqslant m_k$,*

$$s_k(n) = \begin{cases} 6n - 12 & \text{if } k < 6, \\ 6n - 24 \text{ (resp., } 6n - 27) & \text{if } k = 6 \text{ and } n \text{ is even (resp., odd)}, \\ W_k(n) & \text{if } k \geqslant 7. \end{cases}$$

We currently require $m_k = 14$ if $k \leqslant 5$, $m_6 = 11$, $m_7 = 71$, $m_8 = 75$, $m_9 = 244$, $m_{10} = 303$, $m_{11} = 92$, and $m_k = 3k - 10$ for $k \geqslant 12$. The remainder of the paper consists of the proof of the theorem.

2. The proof of Theorem 2

By adding edges if necessary, we may assume that a graph that achieves $s_k(n)$ is a simple plane triangulation. After some general remarks about triangulations, the proof will be treated in cases depending on the value of k. Because of the results of West and Will noted above, the cases $k \geqslant 12$ have already been proven, and for the cases $7 \leqslant k \leqslant 11$ it suffices to give constructions achieving the upper bound, $W_k(n)$. The cases $k \leqslant 6$ will follow easily from Theorem 1.

Lemma 1 (Griggs and Lin [1]). *A simple plane triangulation on $n > 3$ vertices has smallest degree at least 3. If $n > 4$, then no two vertices of degree 3 are adjacent.*

Define the *skeleton* G' of a plane triangulation G to be the subgraph of G induced by the vertices of degree > 3. By Lemma 1, if $|V(G)| > 4$, then G' is also a plane triangulation. A face of G' is said to be *empty* if it contains no (degree 3) vertex of G. Given k, an empty face of G' is said to be *distinguished* if every vertex on the boundary has degree at least $k - 1$ in G. In the following constructions, only the skeleton will be given, and $*$ appears in each non-empty face.

One simple observation we shall use several times is as follows: Let $7 \leqslant k \leqslant 11$, and suppose we know that $s_k(n) \geqslant W_k(n+1) - 3$. Further suppose that there is a graph G achieving $s_k(n)$ such that G' has a distinguished empty face. Then,

$$s_k(n+1) = s_k(n) + 3 = W_k(n+1). \tag{1}$$

This follows from the hypotheses by inserting a vertex into the distinguished face of G'. In fact, if $s_k(n) = W_k(n)$ and for some $i > 0$, $W_k(n+i) = W_k(n) + 3i$ and there is G achieving $s_k(n)$ such that G' has at least i distinguished empty faces, then

$$s_k(n+1) = W_k(n+1), \ldots, s_k(n+i) = W_k(n+i).$$

One class of graphs introduced in [1] that is important to us consists of the simple plane triangulations which achieve the bound $a_k(n)$. We say such a graph is *optimal* for $a_k(n)$. If a graph G is optimal for $a_k(n)$ with $6 \leqslant k \leqslant 11$ and if the lower bound $T'_k(n)$ is an integer, then we say G is *ideal* for k. In [1] it is shown that a simple plane triangulation G is ideal for k, $6 \leqslant k \leqslant 10$, if each vertex has degree 3 or degree k; G is ideal for $k = 11$ if there are 6 vertices of degree 4 and the remaining vertices have degree 3 or degree 11. Denote an ideal graph for k on n vertices by $\mathrm{IG}(k, n)$.

We say a graph G is *extreme* for k if it achieves the bound $W_k(n)$ given by West and Will, and such a graph will be denoted by $\mathrm{EG}(k, n)$. Note that ideal graphs are extreme graphs.

Our main method of constructing the required infinite families of extreme graphs is the repeated insertion of layers. A *layer of length t* in a triangulation consists of two nested t-cycles with a $2t$-cycle alternating between the two t-cycles. There are $2t$ triangular faces in the layer, some of which may contain $*$'s. We seek to expand an extreme graph by replacing a layer by two or more nested layers of the same length such that the inner and outer layers of the replacement have stars in the same sequence as the original single layer. Then the portions of the original graph both outside and inside the layer can be rotated to properly align with the new nested layers that replace the original layer.

We now proceed with the cases of the proof.

2.1. The case $k \leqslant 5$

As implied by Theorem 1, for $n \geqslant 14$ there exist plane triangulations G on n vertices such that every vertex has degree at least 5. Since G has $3n - 6$ edges, it follows that $s_k(n) = 6n - 12$ for $n \geqslant 14$.

2.2. The case $k = 6$

Theorem 1 notes that for $n \geqslant 8$, every plane triangulation G on n vertices has at least 4 vertices of degree < 6. By Lemma 1, all vertices in G have degree at least 3. It follows that $s_6(n) \leqslant 6n - 24$. The upper bound, $6n - 24$, is achieved for even $n \geqslant 8$ by Theorem 1, which implies the existence of $\mathrm{IG}(6, n)$.

Similarly, for odd $n \geqslant 8$, Theorem 1 means that G must have at least 5 vertices of degree < 6 (this is the result of [2]). For odd $n \geqslant 11$, the graph $\mathrm{IG}(6, n-1)$ will have an empty face, and if we insert a vertex into the empty face, adjacent to the three vertices on its boundary, we achieve the bound, $6n - 27$, on $s_6(n)$.

2.3. The case $k = 7$

Fig. 1(a) shows an extreme graph $\mathrm{EG}(7, 34)$ in which the two thick 3-cycles induce a layer of length 3. This layer can be replaced by two nested layers of Fig. 1(b), which adds four total vertices, one of which has degree 3. Repeatedly inserting a layer in this way gives a sequence of extreme graphs $\mathrm{EG}(7, 34 + 4i)$, $i \geqslant 0$. In Figs. 1(c) and (d) are $\mathrm{EG}(7, 39)$ and $\mathrm{EG}(7, 59)$, each with a layer of length 6. Figs. 1(e) and (f) show that the layer can be replaced by inserting two or three new layers, respectively, adding 16 or 24 new vertices, respectively. Repeated insertion of layers to these two graphs yields a sequence of extreme graphs $\mathrm{EG}(7, 71 + 4i)$, $i \geqslant 0$. For $n = 36 + 4i$, $i \geqslant 0$, ideal graphs $\mathrm{IG}(7, n)$ have been found in [1], and ideal graphs are extreme. There are distinguished empty faces in the skeletons of these ideal graphs. By (1), $s_7(37 + 4i) = W_7(37 + 4i)$, $i \geqslant 0$. To sum up, we get $s_7(n) = W_7(n)$ if $n \geqslant 71$.

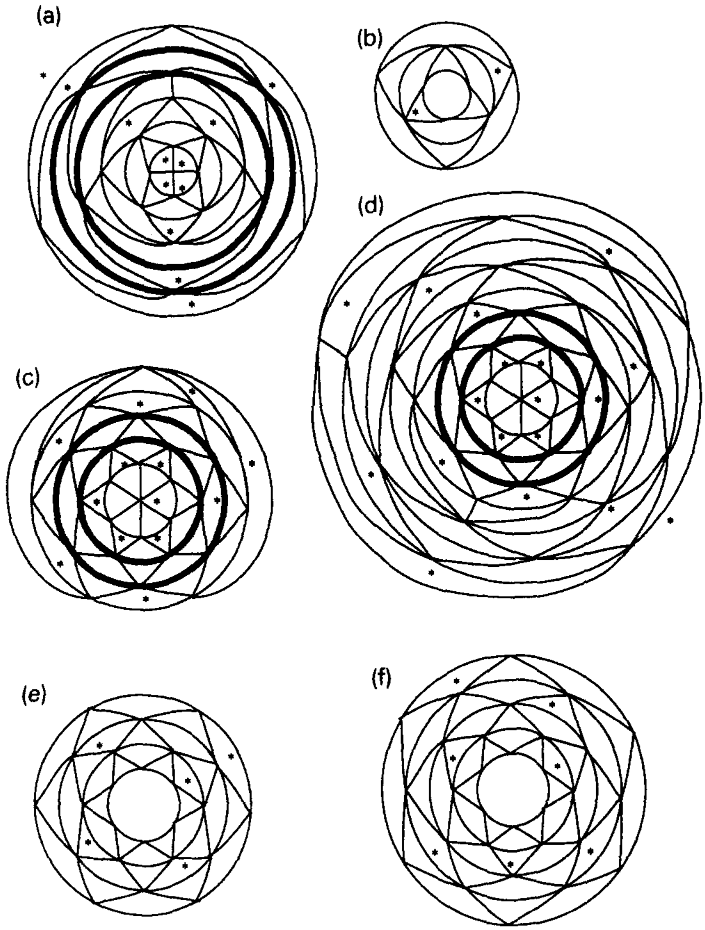

Fig. 1.

2.4. The case $k = 8$

For $n = 24 + 5i$, $i \geqslant 0$, ideal graphs IG$(8, n)$ have been constructed in [1], which takes care of the case $n \equiv 4 \pmod 5$. Further, their skeletons contain distinguished empty faces. Since $W_8(n + 1) = W_8(n) + 3$ for $n \equiv 4 \pmod 5$, we obtain from (1) that $s_8(25 + 5i) = W_8(25 + 5i)$, $i \geqslant 0$, which takes care of $0 \pmod 5$. Next we treat $3 \pmod 5$. Fig. 2(a) is an EG(8, 53) with a layer of length 6. Fig. 2(b) shows that the layer can be repeated by inserting a layer. Thus, we get a sequence of extreme graphs EG$(8, 53 + 10i)$, $i \geqslant 0$. Fig. 2(c) is an EG(8, 58) with a layer similar to that of Fig. 2(a). Therefore, we have a sequence of extreme graphs EG$(8, 58 + 10i)$, $i \geqslant 0$. For the case $1 \pmod 5$, Fig. 2(d) is an EG(8, 51) with a layer of length 6. Fig. 2(e) shows that the layer can be repeated by inserting a layer. Hence, a sequence of extreme graphs

EG(8, 51 + 10i) follows, $i \geqslant 0$. Fig. 3(a) is an EG(8, 56) with a layer of length 6. Figs. 3(b) and (c) show that the layer can be repeated by inserting two or three layers, and we get a sequence of extreme graphs EG(8, 56 + 20i + 30j), $i, j \geqslant 0$. These graphs for 1 (mod 5) have a distinguished empty face. As before, we can check that (1) applies

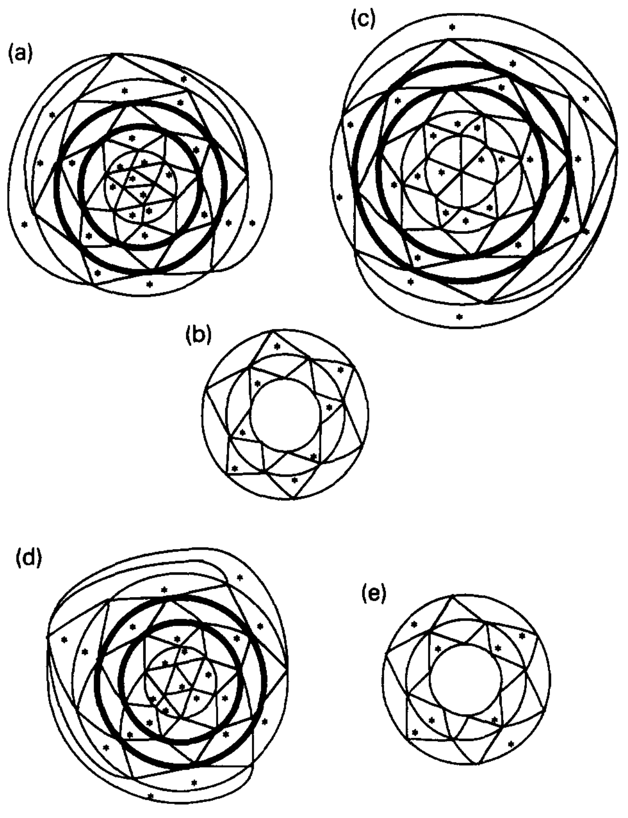

Fig. 2.

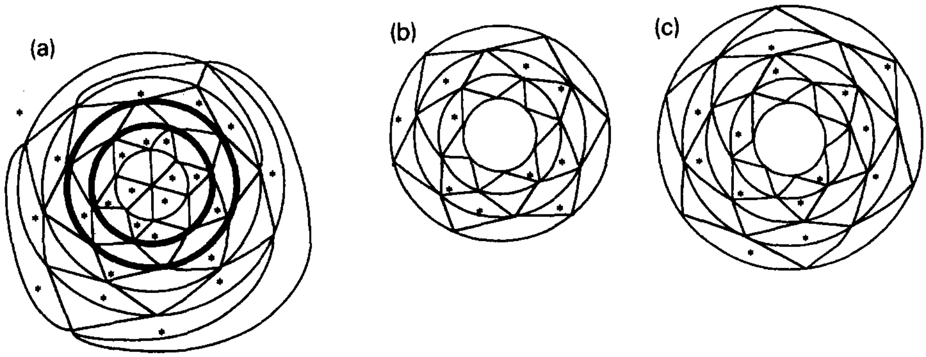

Fig. 3.

to give us the 2 (mod 5) case. We conclude from all the cases that $s_8(n) = W_8(n)$ if $n \geqslant 75$.

2.5. The case $k = 9$

For even $n \geqslant 244$, ideal graphs $\mathrm{IG}(9, n)$ have been constructed in [1], and their skeletons contain distinguished empty faces. Hence, by (1), $s_9(n) = W_9(n)$ if $n \geqslant 244$.

2.6. The case $k = 10$

We deal with cases in n (mod 7), beginning with $n \equiv 4$. For $n = 53 + 7i$, $i \geqslant 0$, ideal graphs $\mathrm{IG}(10, n)$ have been constructed in [1], and their skeletons contain more than

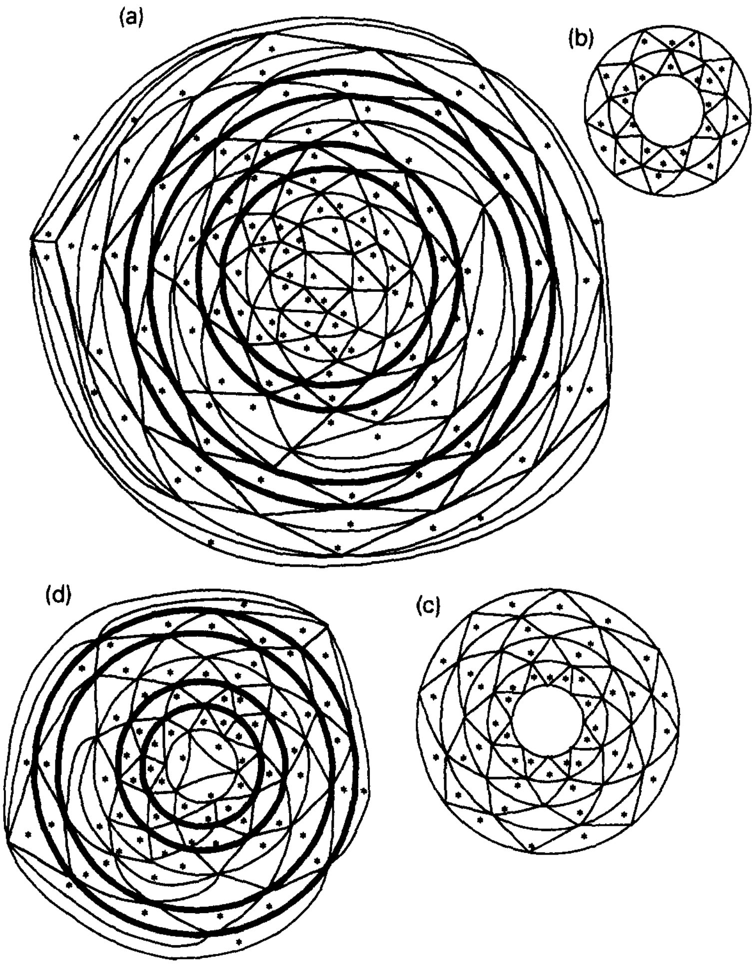

Fig. 4.

one distinguished empty face. One checks for $n \equiv 4 \pmod 7$ that

$$W_{10}(n+2) = W_{10}(n+1) + 3 = W_{10}(n) + 6,$$

so that we can apply (1) to the cases $n \equiv 5, 6 \pmod 7$: $s_{10}(54+7i) = W_{10}(54+7i)$ and $s_{10}(55+7i) = W_{10}(55+7i)$, $i \geqslant 0$. For the case 2 (mod 7), Fig. 4(a) is an EG(10, 205) with layers of lengths 9 and 8. Figs. 4(b) and (c) show that they can be repeated by inserting one layer and three layers, respectively. Hence, we get a sequence of extreme graphs EG$(10, 205+21i+56j)$, $i, j \geqslant 0$. Thus, $s_{10}(303+7i) = W_{10}(303+7i)$. It can be checked in the usual way that (1) applies to deal with $3 \pmod 7$: $s_{10}(304+7i) = W_{10}(304+7i)$, $i \geqslant 0$. Now consider $n \equiv 0 \pmod 7$. Fig. 4(d) is an EG(10, 126) with two layers similar to those of Fig. 4(a), so we get a sequence of extreme graphs EG$(10, 126+21i+56j)$, $i, j \geqslant 0$. Therefore,

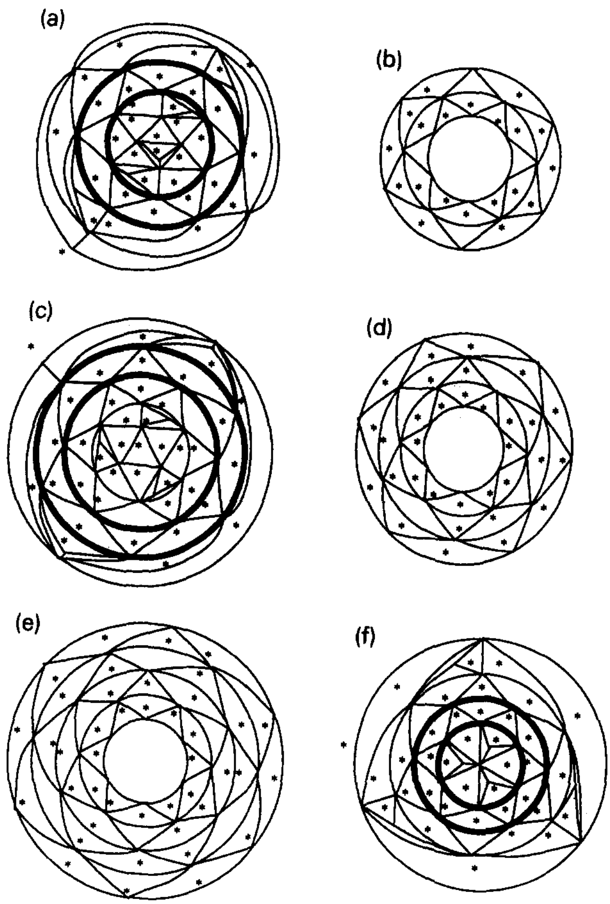

Fig. 5.

$s_{10}(224 + 7i) = W_{10}(224 + 7i)$. Finally we can apply (1) as usual to resolve $n \equiv 1 \pmod 7$: $s_{10}(225 + 7i) = W_{10}(225 + 7i)$, $i \geqslant 0$. To sum up, we have $s_{10}(n) = W_{10}(n)$ for $n \geqslant 303$.

2.7. The case $k = 11$

For $n = 54 + 8i$, $i \geqslant 0$, ideal graphs $\mathrm{IG}(11, n)$ have been constructed in [1], and each has more than one distinguished empty face. By (1), $s_{11}(55 + 8i) = W_{11}(55 + 8i)$, and $s_{11}(56 + 8i) = W_{11}(56 + 8i)$, $i \geqslant 0$. Fig. 5(a) is an $\mathrm{EG}(11, 60)$ with a layer of length 6. Fig. 5(b) shows that the layer can be repeated by inserting one layer. Hence, we have a sequence of extreme graphs $\mathrm{EG}(11, 60 + 16i)$, $i \geqslant 0$. Fig. 5(c) is an $\mathrm{EG}(11, 68)$ with a layer of length 6. Figs. 5(d) and (e) show that the layer can be repeated by inserting two or three layers. Hence, we get a sequence of extreme graphs $\mathrm{EG}(11, 68 + 32i + 48j)$, $i,j \geqslant 0$. Thus, $s_{11}(92 + 8i) = W_{11}(92 + 8i)$, and by (1),

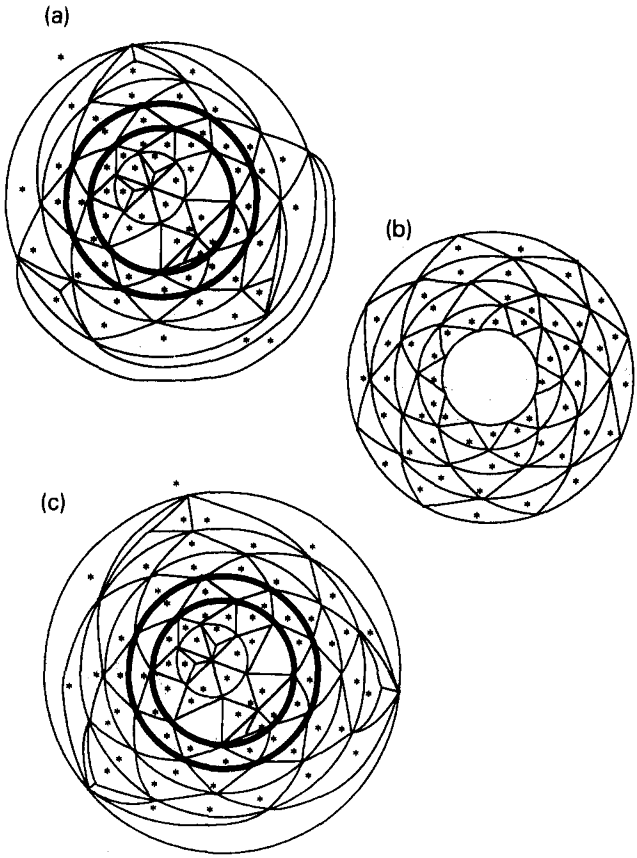

Fig. 6.

$s_{11}(93 + 8i) = W_{11}(93 + 8i)$, $i \geq 0$. Fig. 5(f) is an EG(11,65), with a layer similar to that of Fig. 5(c), and this gives a sequence of extreme graphs EG$(11, 65 + 32i + 48j)$, $i,j \geq 0$. Fig. 6(a) is an EG(11,105) with a layer of length 8. Fig. 6(b) shows that the layer can be repeated by inserting three layers. Hence, we have a sequence of extreme graphs EG$(11,105 + 64i)$, $i \geq 0$. Fig. 6(c) is an EG(11,121) with a layer similar to that of Fig. 6(a), giving a sequence of extreme graphs EG$(11,121 + 64i)$, $i \geq 0$. Fig. 7(a) is an EG(11,137) with a layer of length 8. Fig. 7(b) shows that the layer can be repeated by inserting three layers, and so we have a sequence of extreme graphs EG$(11,137 + 64i)$, $i \geq 0$. Fig. 7(c) is an EG(11,153) with a layer of length 8. Fig. 7(d) shows that the layer can be repeated by inserting three layers, and we have a sequence of extreme graphs EG$(11,153 + 64i)$, $i \geq 0$. Therefore, $s_{11}(97 + 8i) = W_{11}(97 + 8i)$, $i \geq 0$. Note that the EG's have more than one distinguished empty face. By (1), $s_{11}(98 + 8i) = W_{11}(98 + 8i)$ and $s_{11}(99 + 8i) = W_{11}(99 + 8i)$, $i \geq 0$. To sum up, $s_{11}(n) = W_{11}(n)$ if $n \geq 92$. □

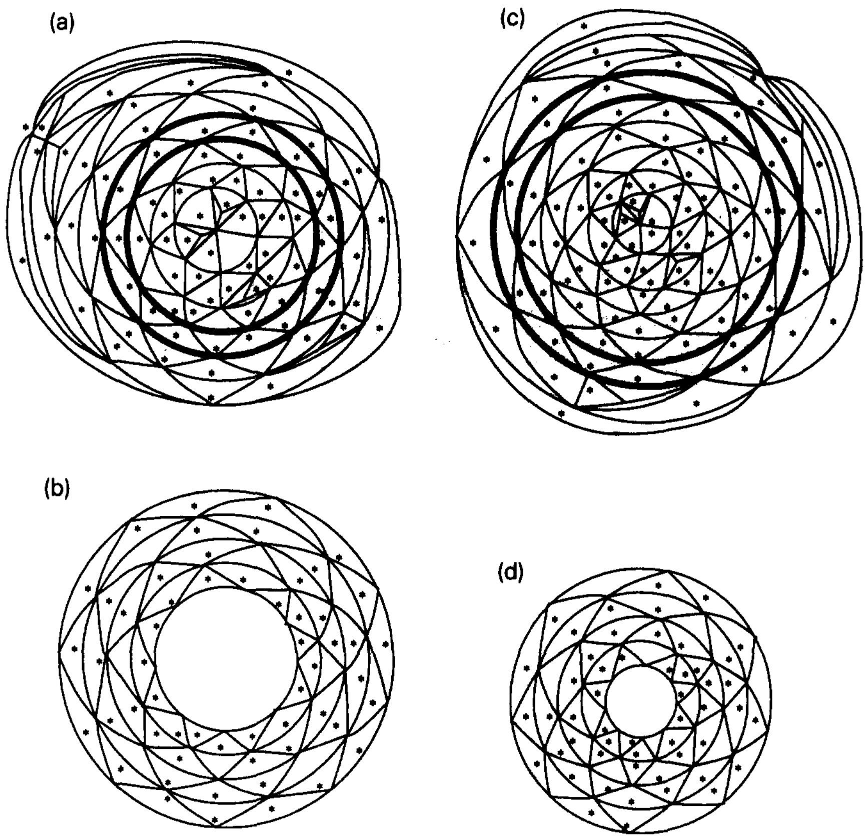

Fig. 7.

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